REPORT No. 497

COMPUTATION OF THE TWO-DIMENSIONAL FLOW IN A LAMINAR BOUNDARY LAYER

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SUMMARY

A comparison is made of the boundary-layer flow computed by the approximate method developed by K. Pohlhausen with the exact solutions which have been published for several special cases. A modification of Pohlhausen's method has been developed which extends the range of application at the expense of some decrease in the accuracy of the approximation.

The work was carried out at the National Bureau of Standards, in part with the cooperation and financial assistance of the National Advisory Committee for Aeronautics.

INTRODUCTION

The concept of the boundary layer introduced in modern aerodynamics by Prandtl has been extraordinarily fruitful in the interpretation of experimental data. As yet, it is not possible to make the interpretations quantitative, except in a few instances, since the equations describing the flow are nonlinear, and their mathematical solution is extraordinarily difficult, if not altogether impractical in many cases of interest. Pohlhausen (reference 1) developed an approximate method of solution of the equations for 2-dimensional laminar flow which has been criticised by von Mises (reference 2). Since Pohlhausen's method and related methods are within the mathematical skill of most experimenters, it seemed worth while to study the possibilities and limitations of such methods as judged by the instances for which exact solutions are known.

POHLHAUSEN'S SOLUTION

The equations for the steady laminar flow of an incompressible fluid in the boundary layer along a 2-dimensional surface whose radius of curvature is large as compared with the thickness of the layer are as follows:[1]

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\nu\frac{\partial^2 u}{\partial y^2}-\frac{1}{\rho}\frac{\partial p}{\partial x} \tag{1}$$

$$\frac{\partial p}{\partial y}=0 \tag{2}$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \tag{3}$$

where u is the tangential component of the velocity, v the normal component, x the distance measured along the surface, y the distance measured normal to the surface, ν the kinematic viscosity of the fluid, ρ the density of the fluid, and p the pressure. At the surface, u and v are zero. As y increases, u approaches U, the speed in the potential flow outside the boundary layer, asymptotically. U is a function of x in general. From equation (2) the pressure within the boundary layer is independent of y and equal to that in the potential flow. Since in the potential flow $p+\frac{1}{2}\rho U^2$ is constant,

$$\frac{\partial p}{\partial x}=-\rho UU' \tag{4}$$

$$U' \text{ denoting } \frac{\mathrm{d}U}{\mathrm{d}x}.$$

From equations (1) to (4), an important integral equation may be derived (reference 1), namely, the Kármán integral relation

$$-2U'\int_0^\infty (U-u)\mathrm{d}y-U\frac{\mathrm{d}}{\mathrm{d}x}\int_0^\infty (U-u)\mathrm{d}y + \frac{\mathrm{d}}{\mathrm{d}x}\int_0^\infty (U-u)^2\mathrm{d}y=-\nu\left(\frac{\partial u}{\partial y}\right)_{y=0} \tag{5}$$

In Pohlhausen's approximate method of solution, a suitable assumption is made as to the shape of the velocity-distribution curve, leaving undetermined a parameter δ which may be regarded as the "thickness" of the boundary layer. δ is then determined as a function of x from the relation (5), following which the velocity distribution itself may be computed. The procedure is reviewed here, omitting the algebraic manipulations which are straightforward, though tedious.

Pohlhausen assumes

$$u=ay+by^2+cy^3+dy^4 \tag{6}$$

To determine the 4 coefficients, 4 conditions are necessary. It is first required that the distribution within the boundary layer be continuous both as to magnitude

[1] A full discussion of the approximations made in deriving the boundary-layer equations is given by K. Hiemenz in Dinglers Polytechnische Journal, vol. 326, 911, p. 321.

and slope with the potential flow at $y=\delta$, where δ is a function of x, at present undetermined. This gives the 2 conditions:

$$u=U \text{ at } y=\delta \tag{7}$$

$$\frac{\partial u}{\partial y}=0 \text{ at } y=\delta \tag{8}$$

The distribution is then made to satisfy the differential equation (1) at the two boundaries. This requires [2]

$$\frac{\partial^2 u}{\partial y^2}=\frac{1}{\rho\nu}\frac{\partial p}{\partial x}=-\frac{UU'}{\nu} \text{ at } y=0 \tag{9}$$

and

$$\frac{\partial^2 u}{\partial y^2}=0 \text{ at } y=\delta \tag{10}$$

From the 4 conditions 7, 8, 9, 10, the 4 coefficients a, b, c, d in (6), may be determined. It is found that

$$a=\left(2+\frac{\lambda}{6}\right)\frac{U}{\delta};\ b=\left(-\frac{\lambda}{2}\right)\frac{U}{\delta^2};\ c=\left(-2+\frac{\lambda}{2}\right)\frac{U}{\delta^3};$$

$d=\left(1-\frac{\lambda}{6}\right)\frac{U}{\delta^4}$; that is, equation (6) is of the form

$$\frac{u}{U}=(a_0+a_1\lambda)\frac{y}{\delta}+(b_0+b_1\lambda)\frac{y^2}{\delta^2}+(c_0+c_1\lambda)\frac{y^3}{\delta^3}+(d_0+d_1\lambda)\frac{y^4}{\delta^4} \tag{11}$$

where $\lambda=\frac{U'\delta^2}{\nu}$ and a_0, a_1, etc., are numbers having the values $a_0=2$, $a_1=1/6$, $b_0=0$, $b_1=-1/2$, $c_0=-2$, $c_1=1/2$, $d_0=1$, $d_1=-1/6$. The parameter λ gives the influence of the potential flow on the shape of the curve relating the nondimensional quantities u/U and y/δ. If λ is constant (or zero), the distribution curves are homologous for all values of x, i.e., the curve of u/U vs. y/δ is independent of x; δ is, however, a function of x. If $a_0+a_1\lambda$ is negative, u/U is negative for small values of y/δ, indicating a reverse flow near the surface. The criterion for the beginning of reverse flow (separation) is that

$$\lambda=\frac{U'\delta^2}{\nu}=-\frac{a_0}{a_1} \tag{12}$$

The value of δ is determined from (5), a procedure which amounts to satisfying the differential equation on the average and at the boundaries (by 9 and 10) but not at every point. With the approximation (11) for the distribution within the layer, the upper limits of the integrals in (5) may be taken as δ instead of ∞; since at values of $y/\delta>1$, u is assumed equal to U and $U-u$ vanishes.

From (11) it may be shown that

$$\int_0^\delta (U-u)\,\mathrm{d}y=U\delta\,(S+T\lambda) \tag{13}$$

where S and T are numbers computed from a_0, a_1, etc.

Noting that both δ and λ are functions of x and that

$$\lambda=\frac{U'\delta^2}{\nu};\ \frac{\mathrm{d}\lambda}{\mathrm{d}x}=\frac{2U'\delta}{\nu}\frac{\mathrm{d}\delta}{\mathrm{d}x}+\frac{U''\delta^2}{\nu}=\frac{2\lambda}{\delta}\frac{\mathrm{d}\delta}{\mathrm{d}x}+\frac{U''\lambda}{U'} \tag{14}$$

where U'' denotes $\frac{\mathrm{d}^2U}{\mathrm{d}x^2}$,

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_0^\delta (U-u)\,\mathrm{d}y=(S+3T\lambda)U\frac{\mathrm{d}\delta}{\mathrm{d}x}+T\frac{UU''}{U'}\delta\lambda+U'\delta(S+T\lambda) \tag{15}$$

Likewise $$\int_0^\delta (U-u)^2\,\mathrm{d}y=U^2\delta\,(K+L\lambda+M\lambda^2) \tag{16}$$

where K, L, and M are numbers computed from a_0, a_1, etc.,

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_0^\delta (U-u)^2\,\mathrm{d}y=U^2\frac{\mathrm{d}\delta}{\mathrm{d}x}(K+3L\lambda+5M\lambda^2)+2UU'\delta\,(K+L\lambda+M\lambda^2)+U^2\delta\frac{\lambda U''}{U'}(L+2M\lambda) \tag{17}$$

Substitution in (5) gives, writing $\frac{\delta^2}{\nu}=z$

$$\frac{U}{2}\frac{\mathrm{d}z}{\mathrm{d}x}=\frac{-a_0-(2K-3S+a_1)U'z-\left\{2L-3T+(L-T)\frac{UU''}{U'^2}\right\}U'^2z^2-2M\left(1+\frac{UU''}{U'^2}\right)U'^3z^3}{K-S+3\,(L-T)U'z+5MU'^2z^2} \tag{18}$$

The values of S, T, K, L, and M are given by

$$S=1-\frac{a_0}{2}-\frac{b_0}{3}-\frac{c_0}{4}-\frac{d_0}{5}=0.3$$

$$T=-\frac{a_1}{2}-\frac{b_1}{3}-\frac{c_1}{4}-\frac{d_1}{5}=-\frac{1}{120}$$

$$K=1+\frac{a_0^2}{3}+\frac{b_0^2}{5}+\frac{c_0^2}{7}+\frac{d_0^2}{9}-a_0-\frac{2b_0}{3}-\frac{c_0}{2}-\frac{2d_0}{5}+\frac{a_0b_0}{2}+\frac{2a_0c_0}{5}+\frac{a_0d_0}{3}+\frac{b_0c_0}{3}+\frac{2b_0d_0}{7}+\frac{c_0d_0}{4}=\frac{23}{126}$$

$$L=\frac{2a_0a_1}{3}+\frac{2b_0b_1}{5}+\frac{2c_0c_1}{7}+\frac{2d_0d_1}{9}-a_1-\frac{2b_1}{3}-\frac{c_1}{2}-\frac{2d_1}{5}+\frac{a_0b_1}{2}+\frac{a_1b_0}{2}+\frac{2a_0c_1}{5}+\frac{2a_1c_0}{5}+\frac{a_0d_1}{3}+\frac{a_1d_0}{3}+\frac{b_0c_1}{3}+\frac{b_1c_0}{3}+\frac{2b_0d_1}{7}+\frac{2b_1d_0}{7}+\frac{c_0d_1}{4}+\frac{c_1d_0}{4}=-\frac{11}{1512}$$

$$M=\frac{a_1^2}{3}+\frac{b_1^2}{5}+\frac{c_1^2}{7}+\frac{d_1^2}{9}+\frac{a_1b_1}{2}+\frac{2a_1c_1}{5}+\frac{a_1d_1}{3}+\frac{b_1c_1}{3}+\frac{2b_1d_1}{7}+\frac{c_1d_1}{4}=\frac{1}{9072}$$

The complete solution of Pohlhausen is given then by

$$\frac{u}{U}=\left(2+\frac{\lambda}{6}\right)\frac{y}{\delta}-\left(\frac{\lambda}{2}\right)\frac{y^2}{\delta^2}+\left(-2+\frac{\lambda}{2}\right)\frac{y^3}{\delta^3}+\left(1-\frac{\lambda}{6}\right)\frac{y^4}{\delta^4} \tag{19}$$

where $$\lambda=U'z=\frac{U'\delta^2}{\nu}$$

z and hence δ are to be determined from

[2] Note that at $y=0$, $u=v=0$; at $y=\delta$, $u=U$, $\frac{\partial u}{\partial x}=U'$, $\frac{\partial u}{\partial y}=0$.

$$\frac{\mathrm{d}z}{\mathrm{d}x}=\frac{0.8\left[-9072+1670.4\lambda-\left(47.4+4.8\frac{UU''}{U'^2}\right)\lambda^2-\left(1+\frac{UU''}{U'^2}\right)\lambda^3\right]}{U[-213.12+5.76\lambda+\lambda^2]} \tag{20}$$

Equation (20) is of the form

$$\frac{\mathrm{d}z}{\mathrm{d}x}=\frac{P\ (x,\ z)}{Q\ (x,\ z)} \tag{21}$$

and cannot in general be directly integrated. In any particular case, a graphical solution can be made by the isocline method (reference 3) as illustrated later.

MODIFIED POHLHAUSEN METHOD

In the application of Pohlhausen's method to certain types of problems, difficulty arises because $Q(x,\ z)$ vanishes and $\mathrm{d}z/\mathrm{d}x$ becomes infinite. In at least one case in the literature, this behavior has been taken as an indication of early separation of the flow (reference 4), but investigation shows that the singularity is introduced as a consequence of the nature of the assumed velocity distribution and represents a failure of the method. An attempt was therefore made to remove this limitation on the range of application of Pohlhausen's method. The attempt was not completely successful; but, without serious additional complication, the range has been greatly extended, at the expense of some decrease in the accuracy of the approximation.

The modification introduced was the addition of another term in the expression (11) for the velocity distribution, determining the additional constants to avoid the infinite value of $\mathrm{d}z/\mathrm{d}x$, if possible. Thus the velocity distribution was assumed to be

$$\frac{u}{U}=(a_0+a_1\lambda)\frac{y}{\delta}+(b_0+b_1\lambda)\frac{y^2}{\delta^2}+(c_0+c_1\lambda)\frac{y^3}{\delta^3}+(d_0+d_1\lambda)\frac{y^4}{\delta^4}+(e_0+e_1\lambda)\frac{y^5}{\delta^5} \tag{22}$$

Leaving a_0 and a_1 undetermined and applying the conditions 7, 8, 9, and 10, we find

$$b_0=0,\ b_1=-\frac{1}{2},\ c_0=10-6a_0,\ c_1=\frac{3}{2}-6a_1,\ d_0=-15+8a_0,$$

$$d_1=-\frac{3}{2}+8a_1,\ e_0=6-3a_0,\ e_1=\frac{1}{2}-3a_1.$$

Likewise:

$$S=\frac{1}{2}-\frac{a_0}{10}$$

$$T=\frac{1}{120}-\frac{a_1}{10}$$

$$K=(2715-933a_0+104a_0^2)/6930$$
$$L=(281-69a_0-3732a_1+832a_0a_1)/27720$$
$$M=(416a_1^2-69a_1+3)/27720$$

The solution for $\lambda=0$ ($U'=0$) is known, namely, that given by Blasius (reference 5). The value of a_0 was taken as 1.89 to give a good approximation to that solution.

With this value of a_0:

$$Q(x,\ z)=3328.4064-\lambda(1837.44a_1-241.23)-\lambda^2(2080a_1^2-345a_1+15) \tag{23}$$

To prevent $\mathrm{d}z/\mathrm{d}x$ from becoming infinite, $Q(x,\ z)$ must not vanish. Since in the physical problem λ must be a real number, it would be desirable to have the roots of $Q(x,\ z)=0$ imaginary. It proves to be impossible to make the roots of $Q(x,z)=0$ imaginary by any choice of a_1. Calling the roots λ_1 and λ_2, where $\lambda_1>\lambda_2$ and calling the value for separation $\lambda_s\left(=-\frac{a_0}{a_1}\right)$, a_1 was selected to make $\frac{\lambda_2}{\lambda_s}$ as large as possible.[3] The maximum ratio is found for $a_1=0.11$, $\lambda_s=-17.18$, $\lambda_2=-30.89$, $\lambda_1=48.52$. The values for Pohlhausen's solution (21) are $\lambda_s=-12$, $\lambda_1=-17.76$, $\lambda_2=12$. The range of application is thus extended by the modified method. The improvement in range is not accurately indicated by these figures since the values of δ and hence of λ are not strictly comparable;[4] but the improvement is sufficient to deal with problems that cannot be handled by (21).

The modified solution is then

$$\frac{u}{U}=(1.89+0.11\lambda)\frac{y}{\delta}-\left(\frac{\lambda}{2}\right)\frac{y^2}{\delta^2}+(-1.34+0.84\lambda)\frac{y^3}{\delta^3}+(0.12-0.62\lambda)\frac{y^4}{\delta^4}+(0.33+0.17\lambda)\frac{y^5}{\delta^5} \tag{24}$$

where $\lambda=U'z=\frac{U'\delta^2}{\nu}\cdot$ z and hence δ are to be determined from

$$\frac{\mathrm{d}z}{\mathrm{d}x}=\frac{0.8\left[-59051.9+13783.3\lambda-\left(53.93-14.6947\frac{UU''}{U'^2}\right)\lambda^2-\left(1+\frac{UU''}{U'^2}\right)\lambda^3\right]}{U[-1500.63-17.6337\lambda+\lambda^2]} \tag{25}$$

[3] The modified method was developed in connection with the study of a flow in which separation was expected and in which Pohlhausen's solution failed by $\mathrm{d}z/\mathrm{d}x$ becoming infinite. Hence this choice of a_1. The maximum ratio and the corresponding value of a_1 were found by trial, i.e., by substituting various values of a_1 in $Q(x,\ z)=0$ and computing $\frac{\lambda_2}{\lambda_s}$.

[4] A more reliable index of the improvement is given by a consideration of the so-called "Verdrängungsdicke", δ^*, defined by $\delta^*=\int_0^\delta\left(1-\frac{u}{U}\right)\mathrm{d}y$ and the corresponding $\lambda^*=\frac{U'\delta^{*2}}{\nu}\cdot$ For Pohlhausen's solution, $\delta^*=\delta(0.300-0.00833\lambda)$, whence $\lambda^*=\lambda(0.300-0.00833\lambda)^2$. For the modified solution $\delta^*=\delta(0.311-0.00267\lambda)$, whence $\lambda^*=\lambda(0.311-0.00267\lambda)^2$. It is readily shown that for Pohlhausen's solution, $\lambda_s^*=-1.92$, $\lambda_1^*=-3.56$, $\lambda_2^*=0.48$; for the modified solution, $\lambda_s^*=-2.19$, $\lambda_1^*=-4.78$, $\lambda_2^*=1.60$.

APPLICATION TO FLOW IN WHICH $U=kx^m$

POHLHAUSEN'S APPROXIMATION

A problem for which exact solutions have been given by Falkner and Skan (reference 6) is that in which $U=kx^m$ where k and m are numerical constants. For this case $U'=mU/x$, $U''=m(m-1)\,U/x^2$, and $UU''/U'^2=(m-1)/m$, a constant. Equation (20) becomes, on setting $z=\lambda/U'$, $\frac{dz}{dx}=\frac{1}{U'}\frac{d\lambda}{dx}-\frac{U''}{U'^2}\lambda$ and collecting terms

$$-\frac{5}{3m+1}\,x\,\frac{d\lambda}{dx}=\frac{(\lambda-\lambda_1)\,(\lambda-\lambda_2)\,(\lambda-\lambda_3)}{\lambda^2+5.76\lambda-213.12} \tag{26}$$

where λ_1, λ_2, and λ_3 are the three roots of

$$\lambda^3(0.6m+0.2)+\lambda^2(36m+1.92)+\lambda(-213.12-1123.20m)+7257.6m=0 \tag{27}$$

The variables λ and x are separated, and equation (26) may be readily integrated to give [5]

$$(\lambda-\lambda_1)^a(\lambda-\lambda_2)^b(\lambda-\lambda_3)^c=Ax^{\frac{5}{3m+1}}$$

where A is the constant of integration, and a, b, and c are given by the relations

$$a=\frac{\lambda_1^2+5.76\lambda_1-213.12}{(\lambda_1-\lambda_2)\;(\lambda_1-\lambda_3)}$$

$$b=\frac{\lambda_2^2+5.76\lambda_2-213.12}{(\lambda_2-\lambda_3)\;(\lambda_2-\lambda_1)}$$

$$c=\frac{\lambda_3^2+5.76\lambda_3-213.12}{(\lambda_3-\lambda_1)\;(\lambda_3-\lambda_2)}$$

The constant of integration A must be determined by the boundary condition which fixes the value of λ at some known value of x. Thus if $\lambda=\lambda_c$ at $x=1$, $A=(\lambda_c-\lambda_1)^a\,(\lambda_c-\lambda_2)^b(\lambda_c-\lambda_3)^c$ and the general solution is

$$\frac{(\lambda-\lambda_1)^a}{(\lambda_c-\lambda_1)^a}\,\frac{(\lambda-\lambda_2)^b}{(\lambda_c-\lambda_2)^b}\,\frac{(\lambda-\lambda_3)^c}{(\lambda_c-\lambda_3)^c}=x^{\frac{5}{3m+1}}$$

The behavior of the general solution can be traced by somewhat tedious numerical calculations for definite numerical values of λ_c and m. If λ_c is selected equal to λ_1, λ_2, or λ_3, it is obvious that the general solution degenerates to the particular solution $\lambda=\lambda_1$, $\lambda=\lambda_2$, or $\lambda=\lambda_3$, in which λ is constant and independent of x. These are the so-called homologous solutions studied by Falkner and Skan. Since $\lambda=\frac{U'\delta^2}{\nu}=\frac{mU\delta^2}{\nu x}$ for this case, $\delta=\sqrt{\frac{\lambda}{m}\frac{\nu x}{U}}$. The velocity distribution curves are homologous, the solution for any value of m being given by a curve of $\frac{u}{U}$ vs. $y\sqrt{\frac{Um}{\lambda\nu x}}$ or, since $\frac{\lambda}{m}$ is constant for any m, of $\frac{u}{U}$ vs. $y\sqrt{\frac{U}{\nu x}}$. It is convenient to write $\frac{\lambda}{m}=N$, in which case equation (27) becomes

[5] Assuming $m\neq-1/3$, in which case there are only 2 roots of (27).

$$N^3(0.6m^3+0.2m^2)+N^2(1.92m+36m^2)+N(-213.12-1123.20m)+7257.6=0 \tag{28}$$

Since (28) is a cubic in N, there are in general 3 values of N and hence 3 solutions for each value of m. The results can best be visualized from a graph of N vs. m. This graph is shown approximately in figure 1. The computed values are given in table I.

TABLE I.—ROOTS OF EQUATION (28)

m	N_1	N_2	N_3	λ_1 [a]	λ_2	λ_3
2.00	4.148	8.088	−38.64	8.296	16.176	−77.28
1.50	5.205	11.200	−50.30	7.808	16.800	−75.45
1.00	7.052	17.803	−72.26	7.052	17.803	−72.26
0.50	11.210	39.70	−130.50	5.605	19.850	−65.25
0	34.054	infinite	infinite	0	28.13	−37.73
−0.02	37.90	−1,461.45	1,742.70	−0.758	29.229	−34.85
−0.09	75.28	−386.57	210.84	−6.775	34.791	−18.98
−0.0929	80.68	−378.1	191.16	−7.495	35.13	−17.76
−0.10	120.00	−360.0	120.00	−12.000	36.00	−12.00
−0.30	imag.	−14,754.0	imag.	imag.	442.62	imag.
−0.33	imag.	imag.	infinite	imag.	imag.	infinite
−0.50	imag.	imag.	362.3	imag.	imag.	−181.15
−1.00	imag.	imag.	107.9	imag.	imag.	−107.85

[a] $\lambda_1=mN_1$, $\lambda_2=mN_2$, $\lambda_3=mN_3$.

It may be noted that k does not appear in the solution. k may be either positive or negative; equation (28) is the same in either case. If k is negative, U is negative, i.e., directed in the opposite direction to x and the negative values of N must be selected, since z

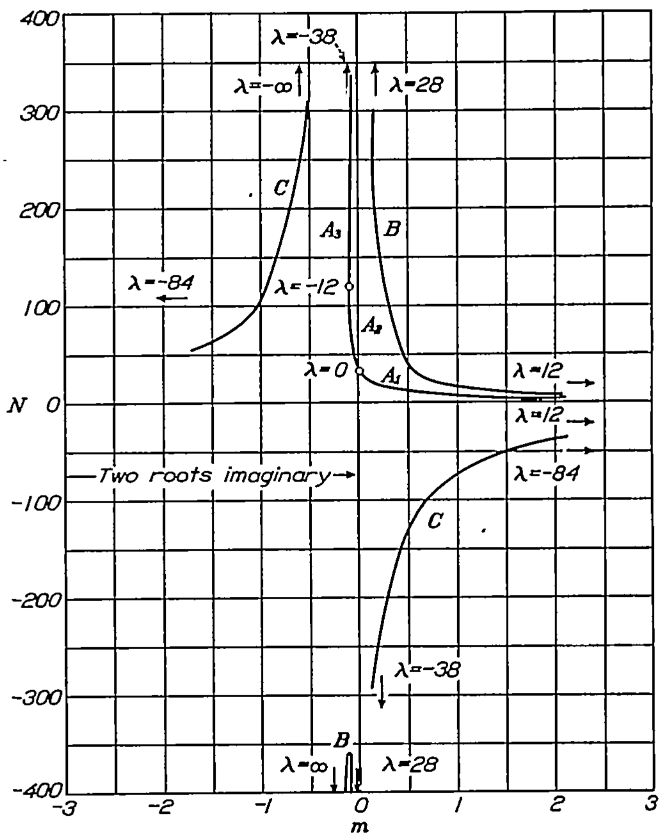

FIGURE 1.—Graph of equation (28), in part schematic

is always a positive quantity, being proportional to the square of the thickness of the boundary layer.

The solution given by Pohlhausen for the case $m=-1$, k negative, is wrong. As shown by figure 1, there is no solution given for this case by Pohlhausen's method. The solution given in reference 1 was apparently obtained by setting $\frac{d(\lambda x^2)}{dx} = -x^2\frac{d\lambda}{dx} + 2\lambda x$, which is obviously wrong.

For the flow $U = kx^m$, no difficulty is encountered because of the vanishing of $Q(x,z)$ in equation (21). λ is constant for a given value of m and when m is such that the corresponding $\lambda(=mN)$ is equal to a root of $Q(x,z)=0$, λ is also a root of $P(x,z)=0$. Hence $\frac{dz}{dx}$ is indeterminate, but not infinite. However, when λ is greater than 12 (one of the roots of $Q(x,z)=0$) the speed within the boundary layer rises to a maximum exceeding U and then falls to U. Although such solutions seem to be possible even in the exact treatment given by Falkner and Skan, they do not, so far as known, occur in any actual flow. The solutions represented by the branches marked B in figure 1 (N_2 and λ_2 in table I) are therefore probably not of interest. Likewise when λ is negative and much greater in absolute value than the other root, -17.76, large negative values of u occur within the boundary layer. The solutions represented by the branches marked C in figure 1 (N_3 and λ_3 in table I) are probably not found in any actual flow. It must be remembered that although such solutions of the boundary-layer equations may exist, the boundary-layer equations are themselves approximations. The solutions represented by branches B and C are of the type which do not conform to the assumptions on which the approximate equations were deduced.

This leaves for consideration the branch labeled A, divided in three sections A_1, A_2, and A_3. A_1 and A_2 represent flows without separation, λ being between $+12$ and -12. The branch A_3 represents a flow with separation. For negative values of m and positive k, representing a flow with velocity decreasing as x increases, if m does not exceed in absolute value 0.10, there are 2 solutions, 1 without separation and 1 with separation. For larger negative values of m, there is either no solution at all or only a solution with separation. This result may be compared with that of Falkner and Skan where a flow without separation was possible if m was negative but not for absolute values greater than 0.09. The branch A_1, represents a flow with a velocity increasing as x increases.

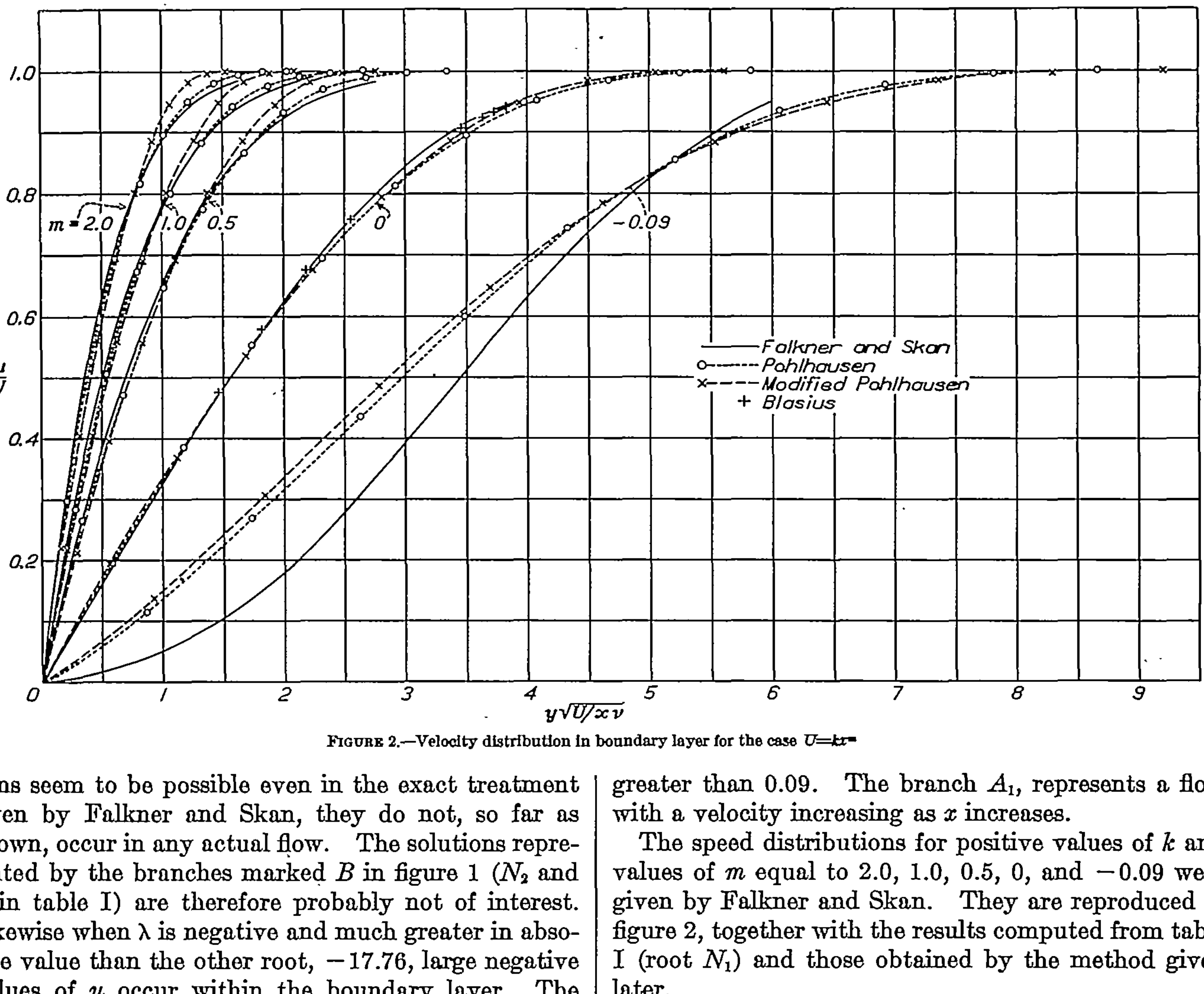

FIGURE 2.—Velocity distribution in boundary layer for the case $U=kx^m$

The speed distributions for positive values of k and values of m equal to 2.0, 1.0, 0.5, 0, and -0.09 were given by Falkner and Skan. They are reproduced in figure 2, together with the results computed from table I (root N_1) and those obtained by the method given later.

The case $m=0$ is that treated by Blasius, whose results are also shown.

The agreement is very close except for $m=-0.09$. For this case the method of Falkner and Skan is probably open to criticism. The series used to represent the solution is not convergent for values of $y\sqrt{\frac{U}{x\nu}}$ much greater than 6, at which value u/U equals 0.95. At $y\sqrt{\frac{U}{x\nu}}=5$, u/U is 0.83 and it appears impossible to tell whether u/U approaches 1 as y increases, since the series is not convergent. Since the approach of u/U to 1 is the criterion for determining the constant which determines the coefficients in the series, it cannot be demonstrated that the solution given is correct. However, this difference may be taken as a warning that Pohlhausen's method may not be satisfactory for negative values of U'.[6]

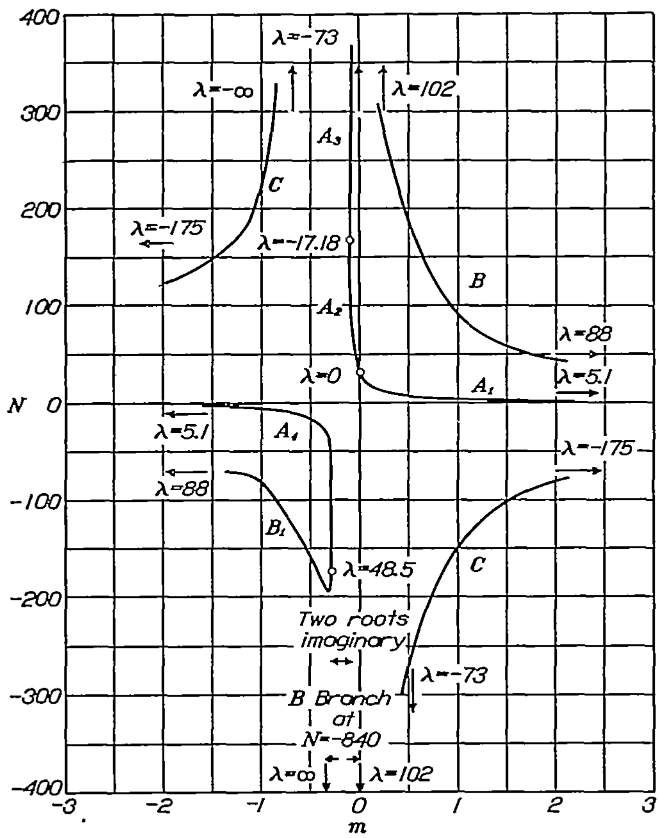

FIGURE 3.—Graph of equation (29), in part schematic

MODIFIED METHOD

The solution by equation (25) proceeds along the same lines as by Pohlhausen's approximation. Equation (28) is replaced by

$$N^3(0.6m^3+0.2m^2)+N^2(49.02194m^2-5.87794m)$$
$$+N(-9526.01m-1500.63)+47241.52=0 \quad (29)$$

The roots are shown graphically in figure 3; the data from which the graph is plotted are given in table II.

[6] Some recent experimental work completed at the National Bureau of Standards indicates that Pohlhausen's approximation is indeed very poor for negative values of U', separation occurring at values of λ of the order of −5.

TABLE II.—ROOTS OF EQUATION (29)

m	N_1	N_2	N_3	λ_1	λ_2	λ_3
2.0	2.352	44.797	−80.065	4.704	89.59	−160.13
1.5	3.056	60.009	−104.067	4.584	90.01	−156.10
1.0	4.365	90.76	−149.06	4.365	90.76	−149.06
0.5	7.638	185.12	−267.29	3.819	92.56	−133.65
0	31.481	Infinite	Infinite	0	102.55	−73.16
−0.09	84.955	377.51	−1,245.6	−7.646	112.10	−33.98
−0.10	117.51	251.93	−1,139.4	−11.751	113.94	−25.10
−0.1036	169.8	169.8	−1,106.9	−17.593	114.69	−17.59
−0.16	Imag.	Imag.	−839.3	Imag.	Imag.	154.28
−0.20	Imag.	Imag.	−852.9	Imag.	Imag.	170.58
−0.30	−43.55	−177.42	−35,086.0	13.065	53.23	10,526.0
−0.33	−33.03	−193.09	Infinite	11.010	64.30	Infinite
−0.50	−15.651	−155.086	778.5	7.826	77.54	−389.3
−1.00	−6.158	−84.25	227.7	6.158	84.25	−227.7

The branches A_1, A_2, A_3, B, and C correspond to similar branches in Pohlhausen's solution. The new feature is the occurrence of branches A_4 and B_1. B_1 is of the B type discussed previously. A_4 gives a solution without separation for negative values of m greater in absolute value than 0.276, if k is negative; i.e., if the flow is one in which the speed increases as x increases. For $m=-1$, the boundary-layer equation may be

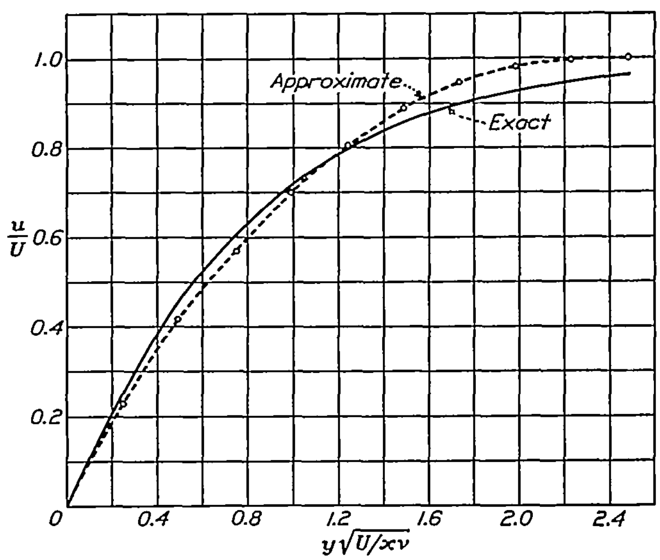

FIGURE 4.—Velocity distribution in boundary layer for the case $U=-k/x$

exactly integrated (reference 1) with the resultant distribution:

$$\frac{-u}{U}=2-3\ \tanh^2\left(1.146+y\sqrt{\frac{-U}{2\nu x}}\right)$$

It should be noted that since k is negative, U is negative, and hence $-U$ is positive. Whereas equation (28) gave no solution, equation (29) gives

$$N=\frac{U\delta^2}{\nu x}=-6.158,\ \lambda=6.158,\ \delta=2.481\sqrt{\frac{\nu x}{-U}}$$

The corresponding velocity distribution from (24) is compared with the exact distribution in figure 4.

The results for m equal to 2.0, 1.0, 0.5, 0, and -0.09 are shown in figure 2 for comparison with the results by Pohlhausen's solution (equation 28) and the results of Falkner and Skan. It is seen that Pohlhausen's solution is in general a better approximation than the modified solution.

In figure 1 and figure 3 the values of λ change continuously along the several branches of the curves. The way in which the branches join at infinity is indicated by the variation of λ. A study of these two figures leads one to believe that equations such as (20) and (25) of the third degree in λ cannot represent the solution over the entire range, and since the degree in λ can be traced to the fact that (19) and (24) are linear in λ, it is probable that the distribution curves either are not linear in λ or that other quantities such as UU''/U'^2 also affect the shape of the distribution curve. Nevertheless, the approximations are valuable where they do give a solution.

APPLICATION TO TRANSVERSE FLOW ABOUT A CYLINDER

The flow in the boundary layer of a cylinder has been computed by J. J. Green (reference 7) by a step-by-step method in which it was assumed that the circumferential velocity is expressible as a power series whose coefficients are functions of the distance along the surface. The approximate solution by equations (24) and (25) was computed for comparison with Green's more nearly exact solution.

While the quantities $U\frac{dz}{dx}$, λ, and $\frac{UU''}{U'^2}$ in equation (25) are nondimensional, z and x are not. As noted by Green, it is convenient to measure all distances in terms of some reference distance, in this case the diameter D of the cylinder, and all speeds in terms of some reference speed, in this case the speed U_0 at a great distance from the cylinder. For simplicity, no new symbols will be introduced, but U is taken to mean the ratio of the speed in the potential flow outside the boundary layer to the reference speed U_0, x the ratio of the distance measured along the surface from the front stagnation point as origin to D, y the ratio of the distance measured normal to the surface to D and δ the ratio of the thickness of the boundary layer to D. It may be shown that the only change required in (25) is in the definition of z which becomes $z=R\delta^2$ where R is the Reynolds Number U_0D/ν; in other words ν is replaced by $1/R$.

The values of U, U', and U'' were taken from data given in Green's paper, table III, columns 4, 5, and 8. The relations between U, U', and U'' and Green's f_1, f_1', and $\bar{q}R$ are as follows:

$$U=\bar{q}$$

$$U'=-\frac{f_1}{R^2\bar{q}}$$

$$\frac{UU''}{U'^2}=-\frac{R^2\bar{q}^2f_1'}{f_1^2}-1$$

The solution of equation (25) was carried out by the isocline method. The first step in this method is the computation of values of $\frac{dz}{dx}$ for several values of z and x. The values of x, U, U', and $\frac{UU''}{U'^2}$ are given in table III, together with the values of $\frac{dz}{dx}$ for several values of z obtained by substitution in equation (25).

TABLE III.—DATA FOR SOLUTION OF EQUATION (25) FOR CYLINDER

θ	x	U	U'	$\frac{UU''}{U'^2}$	$\frac{dz}{dx}$ for $z=$					
					0	0.75	1.50	2.25	3.00	3.75
°										
0	0	0	3.892	−0.2635	+∞	+∞	−∞	−∞	−∞	−∞
5	0.0436	0.149	3.636	+.0445	212.0	76.5	−49.4	−169.0	−286.0	
10	.0872	.307	3.816	−.0270	102.7	34.0	−29.86	−90.4	−150.0	−209.0
15	.1308	.485	3.643	−.0681	65.0	23.4	−15.30	−52.0		
20	.1744	.633	3.535	−.2745	49.9	18.81	−9.90	−37.3	−64.0	−90.6
25	.2180	.789	3.243	−.4235	39.9	17.07	−4.21	−24.5	−44.2	
30	.2616	.928	3.018	−.4815	33.9	15.83	−1.13	−17.29	−33.0	−48.4
35	.3052	1.055	2.845	−.4486	29.8	14.79	0.66	−12.81	−25.88	
40	.3488	1.175	2.699	−.6387	26.79	13.97	1.89	−9.63	−20.79	−31.8
45	.3924	1.283	2.502	−1.4204	24.54	13.64	3.36	−6.47	−15.96	
50	.4360	1.384	2.113	−3.9477	22.76	14.21	6.14	−1.58	−9.03	−16.34
55	.4796	1.463	1.543	−8.1623	21.51	15.59	9.96	4.58	−.62	−5.67
60	.5232	1.522	.957	−19.468	20.69	17.15	13.77	10.55	7.44	4.44
65	.5668	1.549	.452	−91.04	20.33	18.69	17.19	15.78	14.48	13.26
67	.5838	1.554	.161	−923.0	20.27	19.73	19.34	19.08	18.93	18.93
69	.6015	1.554	−.096	−3,748.0	20.27	25.91	26.73	27.81	29.16	30.77

In addition to x, which is the distance along the surface from the front stagnation point, the azimuthal angle θ is given.

An isocline diagram is prepared from the data in table III, that is, a chart with z as ordinate, and x as abscissa with curves showing the loci of constant values of $\frac{dz}{dx}$. In practice it is convenient to change the scale of z relative to x to give values of $\frac{dz}{dx}$ less than 10. In the present case the use of $z/10$ and $\frac{dz/10}{dx}$ is found desirable. A portion of the isocline diagram is shown in figure 5. The numbers on the curves are the values of $\frac{dz/10}{dx}$. The curves are located by interpolation between the values given in table III. Thus at $x=0.4796$ the isocline $\frac{dz/10}{dx}=1$, $\left(\frac{dz}{dx}=10\right)$ lies at a value of z of 1.49; the isocline $\frac{dz/10}{dx}=1.5$ lies at $z=0.829$, etc.

The solution curves of equation (25) must cross the isoclines with the slope indicated on the isocline, that is, the zero isocline must be crossed horizontally, the isocline labeled 1 at a slope of 45°, etc., as indicated by the short lines in figure 5 crossing the isoclines. The particular solution curve in which we are interested is the one which satisfies the boundary conditions at $x=0$, the front stagnation point. Because $U=0$ at $x=0$, we find here a singularity; there is no true boundary layer right at the stagnation point. The isocline chart shows a singular point at which $\frac{dz}{dx}$ is indeterminate. No matter what value of z is assumed at $x=0$,

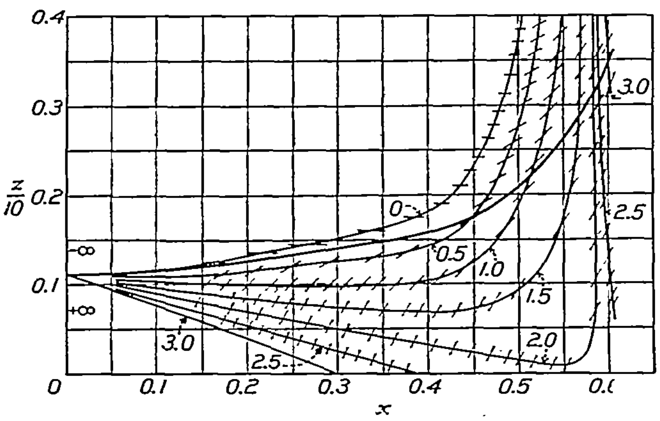

FIGURE 5.—Isocline diagram for boundary layer of a cylinder

the attempt to construct the solution curve leads immediately to the singular point.

It may be shown that the zero isocline leaves the singular point at zero slope and that the desired solution is constructed by starting a solution curve in this manner. The curve is shown in figure 5. In the actual computation, a greater number of isoclines were drawn to a larger scale. From this curve, the values of $z=R\delta^2$ were obtained as a function of x (table IV).

TABLE IV.—VALUES OF $R\delta^2$ FOR CYLINDER

x	$R\delta^2$	x	$R\delta^2$
0	1.125	0.3488	1.500
0.0436	1.125	.3924	1.590
.0872	1.140	.4360	1.731
.1308	1.185	.4796	1.965
.1744	1.233	.5232	2.340
.2180	1.300	.5668	2.880
.2616	1.377	.5838	3.165
.3052	1.433	.6015	3.600

The velocity distribution was then determined from (24). The curves for $x=0.1308$ and $x=0.5668$ only are shown in figure 6. The results of experiments by Green and the results of Green's computation are also shown. At other values of x, the differences are of a similar nature. In Green's computation, the speed in the potential flow was taken from the experimentally measured values, although the pressure distribution was also used in the remainder of the computation. We have used only the results of the pressure distribution.

REMARKS ON THE ACCURACY OF THE APPROXIMATE METHODS

The preceding comparisons show that in these particular cases approximate methods give a fairly good representation of the actual distribution, the differences not exceeding 0.05 U for the modified solution, or 0.02 U for Pohlhausen's solution, where it is applicable. Unfortunately, all of the satisfactory exact solutions are cases in which λ is positive and less than 10 and UU''/U'^2 is small. No satisfactory comparisons are known in which λ is negative.

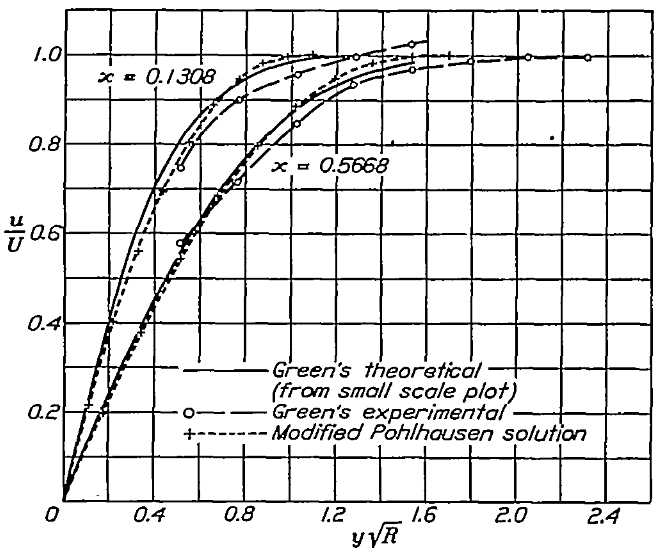

FIGURE 6.—Velocity distribution in boundary layer of cylinder at two distances from the front stagnation point

It is possible to approach the question in a different way. The approximate solution may be used to compute all quantities entering in the differential equation (1) and a check made as to the accuracy with which the equation is satisfied. There will be found a residual error which is most conveniently expressed as a ratio to the last term $\frac{1}{\rho}\frac{\partial p}{\partial x}$ ($=UU'$ by equation 4). This error will be a function of y, nearly zero on the average, exactly zero at $y=0$ and $y=\delta$ and at some intermediate point. Figure 7 shows the maximum positive and negative residuals for values of λ from -20 to $+20$ for $\frac{UU''}{U'^2}=0$, $+44$, and -45 for the modified solution (equations 24 and 25).

The maximum errors for λ between 0 and 8, $\frac{UU''}{U'^2}=0$, the region in which comparisons with exact solutions are available, are of the order of 0.3 to 0.4 times UU'. (Note that $\lambda=0$ corresponds to $U'=0$, hence error $/UU'$ approaches infinity at $\lambda=0$). The errors for negative λ are slightly greater than for positive λ. As UU''/U'^2 increases, the errors become larger, especially for positive values of λ.

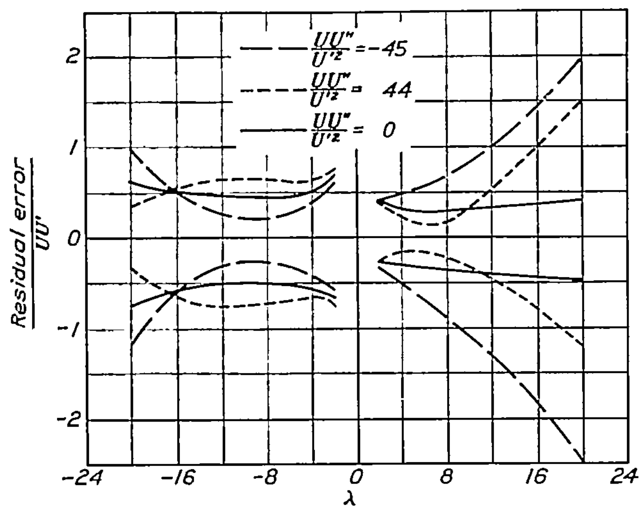

FIGURE 7.—Errors in the modified solution (equations (24) and (25)) at various values of λ and UU''/U'^2

CONCLUSION

Approximate methods of the type suggested by Pohlhausen for calculating the flow in a laminar boundary layer are useful in giving one a fair picture of the flow when the parameters λ and UU''/U'^2 are not too large. The solution given by Pohlhausen fails when U' is positive and large, such that $\lambda=12$. An extension of the range of application of the solution has been accomplished by a modification of Pohlhausen's method with a decrease in the accuracy of the approximation. Comparisons have been made of the approximate solutions with exact solutions for the cases in which exact solutions have been published. The approximate solutions of the type studied appear to be very poor when λ is negative.

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